数理解析研究所講究録
960巻 1996年 118-124

NOTE ON QUESTIONS OF ANDERSON'S *

松田隆輝（Ryûki MATSUDA）

Faculty of Science, Ibaraki University

Let G be a torsion-free abelian (additive) group and S a subsemigroup $\supsetneqq \{0\}$ of G. Then S is called a grading monoid. A non-empty subset I of S is called an ideal of S if $S + I \subset I$. If $I = S + x = (x)$ for some $x \in S$, then I is called a principal ideal of S. If $I + J_1 = I + J_2$ (for ideals J_i) implies $J_1 = J_2$, then I is called a cancellation ideal of S.

Let D be an integral domain and I an ideal of D. If $I = Dx = (x)$ for some $x \in D$, then I is called a principal ideal of D. If $IJ_1 = IJ_2$ (for ideals J_i) implies $J_1 = J_2$, then I is called a cancellation ideal of D.

A domain with a unique maximal ideal is called a local domain.

[AA]D.D.Anderson-D.F.Anderson, Math. Japon 29(1984);
posed the following,

(D)([AA]). Is a cancellation ideal of a local domain principal?

We may pose

(S)(A semigroup version of (D)). Is a cancellation ideal of a grading monoid S principal?

I proved the following,

Theorem 1. Every cancellation ideal of S is principal.

$\mathrm{q}(S) = \{x - y \mid x, y \in S\}$ is called the quotient group of S. If $n\alpha \in S$ ($n \in \mathbf{N}$, $\alpha \in \mathrm{q}(S)$) implies $\alpha \in S$, then S is called integrally closed.

For subsets I_1, I_2 and I of $q(S)$, we put $I_1 :_I I_2 = \{x \in I \mid x + I_2 \subset I_1\}$. And we put $S :_{q(S)} I = I^{-1}$ and $(I^{-1})^{-1} = I^v$. We have, for an ideal

*This is an abstract and the details will appear elsewhere.

$I, S = S^{-1} \subset I^{-1}$ and $S \supset I^v \supset I$.

[A] D.D.Anderson, Comm. in Alg. 16(1988);
posed the following,

(D')([A]). Characterize the integrally closed domains D for which $(A \cap B)^v = A^v \cap B^v$ for all non-zero ideals A, B of D.

We may pose

(S')(A semigroup version of (D')). Characterize the integrally closed S for which $(A \cap B)^v = A^v \cap B^v$ for all ideals A, B of S.

For each maximal ideal M of S, since $S \supset M^v \supset M$, we have either $M^v = M$ or $M^v = S$.

I proved the following,

Theorem 2. (1) Assume that S has a maximal ideal M such that $M^v = M$. If $(A \cap B)^v = A^v \cap B^v$ for all ideals A, B of S, then $I^v = I$ for all ideals I of S.

(2) Assume that for each maximal ideal M of D, $M^v = M$. If $(A \cap B)^v = A^v \cap B^v$ for all non-zero ideals A, B of D, then $I^v = I$ for all non-zero ideals I of D.

Lemma 1. Let $a \in S$. If there exists an ideal which does not contain a, then there exists a maximum ideal which does not contain a.

Proof. Let $\{J_\lambda \mid \lambda\}$ be the set of all ideals which does not contain a. Then $\bigcup_\lambda J_\lambda$ is a desired ideal.

Proof of Theorem 1. Let I be a cancellation ideal. Suppose that I is not principal.

If $I = S$, $I = (0)$ is principal; a contradiction. Hence $I \subsetneqq S = (0)$. Let M be the maximum ideal which does not contain 0. Then $I = I + S \supset I + M$. If $I + S = I + M$, since I is cancellation, we have $S = M$;

a contradiction. Hence $I \supsetneqq I+M$. Choose $I \ni x \notin I+M$. Then $(x) = S + x \subset I$. Since I is not principal, $(x) \subsetneqq I$. Choose $I \ni y \notin (x)$. Put $x + y = a$.

If $a \in (2x)$, $x + y = 2x + s(s \in S)$; $y = x + s \in (x)$; a contradiction. Hence $a \notin (2x)$. Let J be the maximum ideal which does not contain a. Then $2x \in J$.

(1) Let $I \ni b \in (x)$: Then $b+a = x+s+x+y = (s+y)+2x \in I+J$.

(2) Let $I \ni b \notin (x)$: If $a \in (b+y)$, then $x+y = b+y+s$ and $x = b+s$. Since $x \notin I+M, s \notin M$. By the choice of M, $(s) \ni 0$. Then $0 = s + s'$ for some $s' \in S$. Then $b = x - s = x + s' \in (x)$; a contradiction. Hence $a \notin (b+y)$. Therefore $b+y \in J$. Then $b+a = x+(b+y) \in I+J$.

By (1) and (2), we have $I + a \subset I + J$. Hence $a \in J$; a contradiction.

An ideal I of S is called a quasi-cancellation ideal if $I + J_1 = I + J_2$ for finitely generated ideals J_1 and J_2 of S implies $J_1 = J_2$.

Let v be a mapping of a torsion-free abelian (additive) group G to a totally ordered abelian (additive) group. If $v(x+y) = v(x) + v(y)$ for all $x, y \in G$, then v is called a valuation on G. $\{x \in G \mid v(x) \geq 0\}$ is called the valuation semigroup belonging to v.

[SM] Sugatani-Matsuda, Proc. 19th Sympos. on Semigroups, Languages and their related Fields, 1995;
showed that there is a pseudo-valuation semigroup S that is not a valuation semigroup and has a quasi-cancellation ideal which is not a cancellation ideal.

It follows

Remark 1. A quasi-cancellation ideal of S need not be principal.

Lemma 2([SM, Prop. 5]). All propositions of [AA] hold for S.

For example, we have

Lemma 3. Every ideal of S is cancellation if and only if either S is a group or S is a $\mathbf{Z}$-valued valuation semigroup.

Lemma 4. Let A be a flat ideal of S. Then the following conditions are equivalent:

(1) A is cancellation.

(2) A is principal.

(3) A is faithfully flat.

Now we have

Corollary 1. Assume that S has the ascending chain condition on principal ideals (ab. a.c.c.p.). If S has a maximal ideal M which is cancellation, then S is a $\mathbf{Z}$-valued valuation semigroup.

Proof. By Theorem 1, $M = (x)$ is principal. Then $\{(0), (x), (2x), (3x), \cdots\}$ is the set of all ideals of S. By Lemma 3, S is a $\mathbf{Z}$-valued valuation semigroup.

Remark 2. Assume that S is a valuation semigroup and has a maximal ideal which is principal. Then S need not be a $\mathbf{Z}$-valued valuation semigroup.

For example, put $G = \mathbf{Z} \oplus \mathbf{Z}$ with $(1,0) < (0,1)$. Let v be the identity mapping of G to G. Let S be the valuation semigroup belonging to v. Then the principal ideal $M = S + (1,0)$ is a maximal ideal of S.

Remark 3. (1) Let A be an ideal of S. Then A is faithfully flat if and only if A is principal.

(2) A flat ideal of S need not be principal.

For example of (2), put $S = \mathbf{Q}_0$. Then the maximal ideal M of S is flat. But M is not principal.

There is a conjecture ([M]) which says: Almost all propositions in multiplicative ideal theory for D hold for S. And it is usually expected that ideal theory of S is simpler than that of D.

Next, a D-submodule A of $K = q(D)$ is called a fractional ideal of D, if $dA \subset D$ for some non-zero $d \in D$. Let $F(D)$ be the set of all non-zero fractional ideals of D. Let $f(D)$ be the set of finitely generated members of $F(D)$. A mapping $A \longmapsto A^*$ of $F(D)$ to $F(D)$ is called a star-operation on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(D)$:

(1) $(a)^* = (a), (aA)^* = aA^*$;

(2) $A \subset A^*$; if $A \subset B$, then $A^* \subset B^*$;

(3) $(A^*)^* = A^*$.

$A \in F(D)$ is called a $*$-ideal if $A^* = A$. An ideal of D is also called an integral ideal of D. An ideal properly contained in D is called a proper integral ideal of D.

Let I be an ideal of D and let $A \in F(D)$. Then the subset $\{x \in K \mid sx \in A$ for some $s \in D - I\}$ of K is denoted by A_I, where $K = q(D)$.

Lemma 5. Let $*$ be a star-operation on D. Let $\{I_\lambda \mid \lambda\}$ be the set of proper integral $*$-ideals of D. Then we have $\bigcap_\lambda A_{I_\lambda} \subset A^*$ for every $A \in F(D)$.

Proof. Let $0 \neq x \in \bigcap_\lambda A_{I_\lambda}$. Then we have $(A :_D x) \not\subset I_\lambda$ for every λ. Hence $(A^* :_D x) \not\subset I_\lambda$. We have $(A^* :_D x) = x^{-1}A^* \cap D$. It follows that

$$(A^* :_D x)^* = (x^{-1}A^* \cap D)^* \subset (x^{-1}A^*)^* \cap D^* = x^{-1}A^* \cap D = (A^* :_D x).$$

Namely $(A^* :_D x)$ is a $*$-ideal of D. It follows that $(A^* :_D x) = D$. $1 \in (A^* :_D x)$ implies $x \in A^*$. We have proved $\bigcap A_{I_\lambda} \subset A^*$.

Theorem 3. Let $*$ be a star-operation on D. Then the following conditions are equivalent:

(1) $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$.

(2) $(A \cap D)^* = A^* \cap D$ for all $A \in F(D)$.

(3) $(A :_D B)^* = (A^* :_D B^*)$ for all $A \in F(D)$ and $B \in f(D)$,

(4) Let $S = \{I_\lambda \mid \lambda\}$ be the set of proper integral $*$-ideals of D. Then $A^* = \bigcap_\lambda A_{I_\lambda}$ for all $A \in F(D)$.

(5) There is a collection $S = \{I_\lambda \mid \lambda\}$ of proper integral $*$-ideals of D with the property that every proper integral $*$-ideal of D is contained in some I_λ, such that $A^* = \{x \in K \mid (A :_D x) \not\subset I_\lambda$ for every $\lambda\}$ for every

$A \in F(D)$.

Proof. (2)$\Longrightarrow$ (4): By Lemma 5, we have $\bigcap_\lambda A_{I_\lambda} \subset A^*$. Conversely, let $0 \neq x \in A^*$. Then

$(x) = (x) \cap A^* = (x)(D \cap x^{-1}A^*) = (x)(D \cap x^{-1}A)^*$.

Hence $D = (D \cap x^{-1}A)^*$. Therefore $D \cap x^{-1}A \not\subset I_\lambda$ for each λ. It follows $x \in A_{I_\lambda}$ for each λ.

(1)$\Longrightarrow$ (3): Set $B = \sum_i b_i D$. Then we have

$(A^* :_D B^*) = \bigcap_i (b_i^{-1}A^* \cap D) = (\bigcap_i b_i^{-1}A \bigcap D)^* = (A :_D B)^*$.

(5)$\Longrightarrow$ (1): Let $x \in A^* \cap B^*$. There exist $s_\lambda, t_\lambda \in D - I_\lambda$ such that $s_\lambda x \in A, t_\lambda x \in B$ for each λ. Then $t_\lambda s_\lambda x \in A \bigcap B$. Then $s_\lambda x \in (A \cap B)^*$ for each λ. It follows $x \in ((A \cap B)^*)^*$. Hence $x \in (A \cap B)^*$.

Remark 4([M]). [A] holds for S.

Proposition. Theorem 3 holds for S.

The proof of Proposition is a semigroup version of that of Theorem 3.

Corollary 2. (1) Assume that S has a maximal ideal M such that $M^* = M$. If $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$. Then $I^* = I$ for all $I \in F(D)$.

(2) Assume that, for each maximal ideal M of D, $M^* = M$. If $(A \cap B)^* = A^* \cap B^*$ for all $A, B \in F(D)$. Then $I^* = I$ for all $I \in F(D)$.

Proof. (2) Let $\{M_\lambda \mid \lambda\}$ be the set of maximal ideals of D. By Theorem 3, $I^* \subset \bigcap_\lambda I_{M_\lambda} = \bigcap_\lambda ID_{M_\lambda} = I$. Hence $I^* = I$.

The proof of (1) is a semigroup version of that of (2).

Theorem 2 is a special case of Corollary 2.

REFERENCES

[A] D.D. Anderson, Star-operations induced by overrings, Comm. in Alg., 16 (1988), 2535-2553.

[AA] D.D. Anderson and D.F. Anderson, Some remarks on cancellation ideals, Math. Japon. 29(1984), 879-886.

[M] R. Matsuda, Torsion-free abelian semigroup rings IX, Bull. Fac. Sci., Ibaraki Univ. 26(1994), 1-12.

[SM] T. Sugatani and R. Matsuda, Cancellation ideals in semigroups and rings, Proc. 19th Sympos. on Semigroups, Languages and their Related Fields, 1995, 56-59.